

HILBERT COEFFICIENTS AND DEPTH OF THE ASSOCIATED GRADED RING OF AN IDEAL

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ABSTRACT. In this expository paper we survey results proved during the last fifty years that relate Hilbert coefficients $e_0(I)$ and $e_1(I)$ of an \mathfrak{m} -primary ideal I in a Cohen-Macaulay local ring (R, \mathfrak{m}) with depth of the associated graded ring $G(I)$. Several results in this area follow from two theorems of S. Huckaba and T. Marley. These were proved using homological techniques. We provide simple proofs using superficial sequences.

1. Introduction

Throughout these notes, (R, \mathfrak{m}) denotes a Noetherian local ring of dimension d and I denotes an \mathfrak{m} -primary ideal of R . Let $\lambda(M)$ denote length of an R -module M . The Hilbert function $H_I(n)$ of I is defined as $H_I(n) = \lambda(R/I^n)$. It is well-known that $H_I(n)$ is a polynomial function of n of degree d . In other words, there is a polynomial $P_I(x) \in \mathbb{Q}[x]$ such that $H_I(n) = P_I(n)$ for all large n . It is written in terms of the binomial coefficients as:

$$P_I(x) = e_0(I) \binom{x+d-1}{d} - e_1(I) \binom{x+d-2}{d-1} + \cdots + (-1)^d e_d(I)$$

where $e_i(I)$ for $i = 0, 1, \dots, d$ are integers, called the Hilbert coefficients of I . The leading coefficient $e_0(I)$, called the multiplicity of I , is well-understood. However, not much is known about other coefficients.

The associated graded ring of I is defined to be the graded ring $G(I) = \bigoplus_{n=0}^{\infty} I^n/I^{n+1}$. The objective of this paper is to survey known results that link depth of $G(I)$ with linear relations among the Hilbert coefficients $e_0(I)$ and $e_1(I)$ of I . We have chosen two theorems of Huckaba and Marley to illustrate the techniques and results in this area. Moreover these theorems quickly yield, as special cases, several results proved over a period of fifty years. We will provide simple proofs of these results and their consequences.

In section two, we will survey the main results proved about the relationship of depth of $G(I)$ and Hilbert coefficients. In section three, we will

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provide a quick introduction to the theory of reductions of ideals. In section four, we prove the main facts for Hilbert polynomial in a one dimensional Cohen-Macaulay local ring. In section five, a detailed treatment of the theory of superficial elements and sequences is given. This theory is not available in the modern text books in commutative algebra. We have gathered the results useful for our purposes from the Chicago Notes of M. P. Murthy [12], the recent book of C. Huneke and I. Swanson [10] and several papers in this area. In section six, we provide a new and very simple proof, using reductions and superficial elements, of two theorems of Huckaba [8] and Marley [9]. We also include a few of their consequences. For undefined terms in this exposition we refer the reader to [2].

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2. A brief survey

In this section we survey results that link the depth of the associated graded ring $G(I)$ of an \mathfrak{m} -primary ideal I of a Cohen-Macaulay local ring (R, \mathfrak{m}) , and linear relations among the Hilbert coefficients $e_0(I)$ and $e_1(I)$ of I . We refer the reader to an excellent survey article by G. Valla [22] on Hilbert functions of graded algebras and in particular of a Cohen-Macaulay local ring. Perhaps the earliest known result in this direction is due to Northcott [14].

Theorem 2.1 (Northcott, 1960). *Let I be an \mathfrak{m} -primary ideal of a Cohen-Macaulay local ring (R, \mathfrak{m}) with R/\mathfrak{m} infinite. Then*

- (a) $e_0(I) - e_1(I) \leq \lambda(R/I)$.
- (b) $e_1(I) \geq 0$ and equality holds if and only if I is generated by d elements. In this case $e_i(I) = 0$ for $i = 1, 2, \dots, d$ and $G(I)$ is isomorphic to a polynomial ring in d indeterminates over R/I .

As a consequence of Northcott's theorem, we observe that a Cohen-Macaulay local ring (R, \mathfrak{m}) is regular if and only if $e_1(\mathfrak{m}) = 0$. The following theorem of Nagata [13, (40.6)] shows that regularity of (R, \mathfrak{m}) can also be characterized in terms of $e_0(\mathfrak{m})$. Recall that a local ring (R, \mathfrak{m}) is called

unmixed if for each associated prime p of the \mathfrak{m} -adic completion \widehat{R} satisfies $\dim \widehat{R}/p = \dim R$.

Theorem 2.2 (Nagata, 1956). *Let (R, \mathfrak{m}) be an unmixed local ring. Then $e_0(I) = 1$ if and only if $I = \mathfrak{m}$ and R is regular.*

Huneke [7] and Ooishi [16] found conditions under which the equality $e_0(I) - e_1(I) = \lambda(R/I)$ holds.

Theorem 2.3 (Huneke, Ooishi, 1987). *Let (R, \mathfrak{m}) be a Cohen-Macaulay local ring with R/\mathfrak{m} infinite. Then $e_0(I) - e_1(I) = \lambda(R/I)$ if and only if for any minimal reduction J of I , $JI = I^2$. Moreover, when this is the case, $G(I)$ is Cohen-Macaulay, $e_i(I) = 0$ for all $i = 2, 3, \dots, d$ and for all $n \geq 0$,*

$$H_I(n) = P_I(n) = e_0(I) \binom{n+d-1}{d} - e_1(I) \binom{n+d-2}{d-1}.$$

Since $e_0(I) = \lambda(R/J)$ for any minimal reduction J of I , we can restate the Huneke-Ooishi theorem as $e_1(I) = \lambda(I/J)$ if and only if $JI = I^2$. Huckaba [8] and Huckaba-Marley [9] obtained interesting generalization of Huneke-Ooishi theorem.

Theorem 2.4 (Huckaba, 1996). *Let (R, \mathfrak{m}) be a Cohen-Macaulay local ring of dimension d with infinite residue field. Let J be a minimal reduction of I . Then $e_1(I) \leq \sum_{n \geq 1} \lambda(I^n/JI^{n-1})$ and equality holds if and only if $\text{depth } G(I) \geq d - 1$.*

The Cohen-Macaulay property of $G(I)$ was characterized in terms of $e_1(I)$ by Huckaba and Marley [9].

Theorem 2.5 (Huckaba-Marley, 1997). *Let (R, \mathfrak{m}) be a Cohen-Macaulay local ring of dimension d with infinite residue field. Let J be a minimal reduction of an \mathfrak{m} -primary ideal I . Then $e_1(I) \geq \sum_{n \geq 1} \ell(I^n/J \cap I^n)$ and equality holds if and only if $G(I)$ is Cohen-Macaulay.*

We will provide simple proofs of both these theorems using induction on d . One of the crucial tools used in the proofs is the so called *Sally Machine*. We will also provide a new proof of Sally machine due to B. Singh.

Now we turn to another line of research that relates depth of $G(I)$ with Hilbert coefficients. The starting point is an inequality due to Abhyankar [1]. Let $\mu(I)$ denote the minimum number of elements required to generate an ideal I in a local ring (R, \mathfrak{m}) .

Theorem 2.6 (Abhyankar, 1967). *Let (R, \mathfrak{m}) be a Cohen-Macaulay local ring of dimension d . Then*

$$(1) \quad e_0(\mathfrak{m}) \geq \mu(\mathfrak{m}) - d + 1.$$

J. Sally, in a long series of papers, investigated the effect of similar inequalities and equalities on the depth of $G(I)$. First she considered Cohen-Macaulay rings in which (1) is an equality [19]. Such rings are said to have *minimal multiplicity* or *maximal embedding dimension*.

Theorem 2.7 (Sally, 1977). *Let (R, \mathfrak{m}) be a d -dimensional Cohen-Macaulay local ring with infinite residue field. Let J be a minimal reduction of \mathfrak{m} . Then R has minimal multiplicity if and only if $J\mathfrak{m} = \mathfrak{m}^2$. In this case, $G(\mathfrak{m})$ is Cohen-Macaulay and for all $n \geq 0$,*

$$H_{\mathfrak{m}}(n) = P_{\mathfrak{m}}(n) = e_0(\mathfrak{m}) \binom{n+d-1}{d} - e_1(\mathfrak{m}) \binom{n+d-2}{d-1}.$$

Sally's conjecture

We say that a Cohen-Macaulay local ring (R, \mathfrak{m}) has *almost maximal embedding dimension* or *almost minimal multiplicity* if $\mu(\mathfrak{m}) = e_0(\mathfrak{m}) + d - 2$. Such rings have been a subject of investigation since the appearance the paper [20] of J. D. Sally. In this paper, among other things, she proved the following

Theorem 2.8 (Sally, 1980). *Let (R, \mathfrak{m}) be a Gorenstein local ring of positive dimension d and having almost maximal embedding dimension. Then $G(\mathfrak{m})$ is Gorenstein and for all $n \geq 2$,*

$$\lambda(\mathfrak{m}^n / \mathfrak{m}^{n+1}) = e_0(\mathfrak{m}) \binom{n+d-2}{d-1} + \binom{n+d-3}{n}.$$

In the paper [21], Sally studied depth of $G(\mathfrak{m})$ for Cohen-Macaulay local ring (R, \mathfrak{m}) of almost maximal dimension by means of their type. Recall that the type of a Cohen-Macaulay local ring R , denoted by $\text{type}(R)$, is defined to be $\dim_k \text{Ext}^d(k, R)$. For rings of almost maximal dimension, depth of $G(\mathfrak{m})$ is dependent on $\text{type}(R)$.

Theorem 2.9 (Sally, 1983). *Let (R, \mathfrak{m}) be a Cohen-Macaulay local ring of positive dimension d . Let R have almost maximal embedding dimension. Then $\text{type}(R) \leq e_0(\mathfrak{m}) - 2$. If $\text{type}(R) < e_0(\mathfrak{m}) - 2$, then $G(\mathfrak{m})$ is Cohen-Macaulay, R and $G(\mathfrak{m})$ have the same type and for all $n \geq 2$,*

$$\lambda(\mathfrak{m}^n / \mathfrak{m}^{n+1}) = e_0(\mathfrak{m}) \binom{n+d-2}{d-1} + \binom{n+d-3}{n}.$$

In [21], Sally raised the question about depth of $G(\mathfrak{m})$ for a Cohen-Macaulay local ring of almost maximal embedding dimension and maximal type $e_0(\mathfrak{m}) - 2$. This question remained open for several years. It was answered independently by M. E. Rossi and G. Valla in [18] and H-J Wang in [26]. The following theorem summarizes the main results found in [18].

Theorem 2.10 (Rossi-Valla, 1996). *Let (R, \mathfrak{m}) be a Cohen-Macaulay local ring of positive dimension d . Put*

$$G = G(\mathfrak{m}), \quad H_G(n) = \lambda(\mathfrak{m}^n / \mathfrak{m}^{n+1}), \quad \text{and} \quad P_G(z) = \sum_{n=0}^{\infty} H_G(n) z^n.$$

Then the following are equivalent:

- (1) *R has almost maximal embedding dimension.*
- (2) *There is an integer s such that $2 \leq s \leq \mu(\mathfrak{m}) - d + 1$, and*

$$P_G(z) = \frac{1 + (\mu(\mathfrak{m}) - d)z + z^s}{(1 - z)^d}.$$

- (3) *If either of the above conditions holds then $\text{depth } G(I) \geq d - 1$ and G is Cohen-Macaulay if and only if $s = 2$.*

Rossi [17] extended the above theorem partially to all \mathfrak{m} -primary ideals. The condition $e_0(\mathfrak{m}) = \mu(\mathfrak{m}) - d + 2$ has an analogue for \mathfrak{m} -primary ideals. It is easy to see that for an \mathfrak{m} -primary ideal I with a minimal reduction J , in a Cohen-Macaulay local ring (R, \mathfrak{m}) with infinite residue field, $\lambda(I^2/JI) = 1$ if and only if $e_0(I) = \lambda(I/I^2) + (1 - d)\lambda(R/I) + 1$. Rossi proved that If $\lambda(I^2/JI) = 1$ then $G(I) \geq d - 1$ [17].

In [4], Elias generalized the condition $\lambda(I^2/JI) = 1$ further and presented a unified treatment of several theorems. Let J be a minimal reduction of an \mathfrak{m} -primary ideal I in R . We say that I and J satisfy the n th Valabrega-Valla condition VV_n if $J \cap I^n = JI^{n-1}$.

Theorem 2.11 (Elias, 1999). *Let R be a Cohen-Macaulay local ring of dimension $d \geq 1$. Let I be an \mathfrak{m} -primary ideal of R and J be a minimal reduction of I . Let t be positive integer such that*

- (1) *I and J satisfy VV_n for $n = 0, 1, \dots, t$,*
- (2) *$\lambda(I^{t+1}/JI^t) = \delta \leq \min\{1, d - 1\}$.*

Then $d - \delta \leq \text{depth } G(I) \leq d$. If $t \geq e_0(I) - 1$, then $G(I)$ is Cohen-Macaulay.

Sally Modules and depth of $G(I)$

Finally, we discuss the important notion of Sally modules introduced by Vasconcelos in [24]. Let R be a Noetherian ring, and let I be an ideal with a reduction J . The *Rees algebra of an ideal I* , $\mathcal{R}(I)$, is defined to be the graded R -algebra $\bigoplus_{n=0}^{\infty} I^n t^n$ where t is an indeterminate. The Sally module $S_J(I)$ of I with respect to J is the $\mathcal{R}(J)$ -module defined in the exact sequence

$$0 \longrightarrow I\mathcal{R}(J) \longrightarrow I\mathcal{R}(I) \longrightarrow S_J(I) := \bigoplus_{n=0}^{\infty} I^{n+1}/IJ^n \longrightarrow 0.$$

We summarize some basic properties of Sally modules found in [24].

Theorem 2.12. *Let (R, \mathfrak{m}) be a d -dimensional Cohen-Macaulay local ring with infinite residue field. Let I be an \mathfrak{m} -primary ideal and let J be a minimal reduction of I . Then*

- (1) *If $S_J(I) \neq 0$ then its associated primes have height 1. In particular, the its dimension as an $\mathcal{R}(J)$ -module is d .*
- (2) *Let $S = S_J(I) = \bigoplus_{n=0}^{\infty} S_n$. Then For large n ,*

$$H_I(n) = e_0(I) \binom{n+d-1}{d} + (\lambda(R/I) - e_0(I)) \binom{n+d-2}{d-1} - \lambda(S_{n-1}).$$

Hence if $S \neq 0$ then $\lambda(S_n)$ is a polynomial function of degree $d-1$. Let s_i for $i = 0, 1, \dots, d-1$ be the Hilbert coefficients of S . Then $e_1(I) = e_0(I) - \lambda(R/I) + s_0$ and for $i \geq 1$, $e_{i+1}(I) = s_i$.

Vaz Pinto studied the relationship of the Sally module with the depth of $G(I)$. She proved the following interesting result:

Theorem 2.13. *Let (R, \mathfrak{m}) be a Cohen-Macaulay local ring with infinite residue field having positive dimension d . Let J be a minimal reduction of an \mathfrak{m} -primary ideal I . Then the following conditions are equivalent:*

- (1) $s_0 = \sum_{n=1}^{r_J(I)} \lambda(I^{n+1}/JI^n)$.
- (2) S is Cohen-Macaulay.
- (3) $\text{depth } G(I) \geq d-1$.

3. Reductions of ideals

Definition 3.1. *Let R be a Noetherian ring, $J \subseteq I$ be ideals of R . If $JI^n = I^{n+1}$ for some n then J is called a reduction of I . The reduction number $r_J(I)$ of I with respect to J is the smallest n such that $JI^n = I^{n+1}$. The reduction number $r(I)$ of I is the smallest among the reduction numbers $r_J(I)$ where J varies over all minimal reductions of I .*

The notion of reduction of an ideal was introduced by Northcott and Rees in the paper [15]. This paper, now a classic, introduced several other important concepts such as fiber cone of an ideal, analytic spread, analytically independent elements etc. Reductions have played a crucial role in understanding Hilbert coefficients, Rees algebras, fiber cones and associated graded rings of ideals. In this section we prove their basic properties to be used in the later sections.

Proposition 3.2. *Let $J \subseteq I$ be ideals of a Noetherian ring R . Then J is reduction of I if and only if $R[It] = \bigoplus_{n=0}^{\infty} I^n t^n$ is a finite $R[Jt]$ -module.*

Proof. Let J be a reduction of I . Then $(J^k t^k)(I^n t^n) = I^{n+k} t^{n+k}$ for all $k \geq 1$ and for $n \geq r$ for some r . Therefore

$$(2) \quad R[It] = R[Jt] + R[Jt](It) + \cdots + R[Jt](I^r t^r).$$

Hence $R[It]$ is a finite $R[Jt]$ -module. Conversely, let $R[It]$ be a finite $R[Jt]$ -module. Since $R[It]$ is a graded $R[Jt]$ -module, there is a finite set of homogeneous generators of $R[It]$ as an $R[Jt]$ -module. Thus (2) follows for some r . Equate the components of degree $r+1$ on both sides of (2) to get

$$I^{r+1}t^{r+1} = J^{r+1}t^{r+1} + J^r It^{r+1} + \cdots + JI^r t^{r+1}.$$

Thus $JI^r = I^{r+1}$. Hence J is a reduction of I . \square

Corollary 3.3. *Let $K \subseteq J \subseteq I$ be ideals of a Noetherian ring R . Then K is a reduction of I if and only if K is a reduction of J and J is a reduction of I .*

Proof. Let K be a reduction of I . Then $R[It]$ is a finite $R[Kt]$ -module. Hence $R[Jt]$ is a finite $R[Kt]$ -module and $R[It]$ is a finite $R[Jt]$ -module. Hence K is a reduction of J and J is a reduction of I . Conversely let K be a reduction of J and J be a reduction of I . Then $R[Jt]$ is a finite $R[Kt]$ -module and $R[It]$ is a finite $R[Jt]$ -module. Hence $R[It]$ is a finite $R[Kt]$ -module. Thus K is a reduction of I . \square

Proposition 3.4. *Let (R, \mathfrak{m}) be a local ring and let I be an ideal of R . Then an ideal $J \subseteq I$ is a reduction of I if and only if $J + \mathfrak{m}I$ is a reduction of I .*

Proof. As $J \subseteq J + \mathfrak{m}I \subseteq I$, $J + \mathfrak{m}I$ is a reduction of I . Conversely, let $(J + \mathfrak{m}I)I^n = I^{n+1}$. Then $JI^n + \mathfrak{m}I^{n+1} = I^{n+1}$. By Nakayama's Lemma $JI^n = I^{n+1}$. Hence J is a reduction of I . \square

Definition 3.5. *Let I be an ideal of a local ring (R, \mathfrak{m}) . The fiber cone $F(I)$ of I is the graded algebra $R[It]/\mathfrak{m}R[It] = \bigoplus_{n=0}^{\infty} I^n/\mathfrak{m}I^n$. The Krull dimension of $F(I)$, denoted by $\ell(I)$, is called the analytic spread of I . An ideal $J \subseteq I$ is called a minimal reduction of I if $J' \subseteq J$ and J' is a reduction of I then $J = J'$.*

We will prove that minimal reductions exist and if R/\mathfrak{m} is infinite then all minimal reductions of an ideal I require exactly $\ell(I)$ minimal generators.

Proposition 3.6. *Let (R, \mathfrak{m}) be a local ring, I an ideal of R . For $a \in I$, put $a^0 = a + \mathfrak{m}I \in I/\mathfrak{m}I$. Let $J = (a_1, a_2, \dots, a_s) \subseteq I$. Then J is a reduction of I if and only if $(a_1^0, a_2^0, \dots, a_s^0)$ is primary for the maximal homogeneous ideal $F(I)_+$. In particular $\mu(J) \geq \ell(I)$.*

Proof. Let J be a reduction of I . Then there is an r such that $JI^r = I^{r+1}$. Note that $(a_1^0, a_2^0, \dots, a_s^0)_n = JI^{n-1}/\mathfrak{m}I^n$ for $n \geq 1$. Hence $(a_1^0, a_2^0, \dots, a_s^0)_n = F(I)_n$ for $n \geq r+1$. Hence $(a_1^0, a_2^0, \dots, a_s^0)$ is $F(I)_+$ -primary.

Conversely let $(a_1^0, a_2^0, \dots, a_s^0)$ be $F(I)_+$ -primary. Hence there exists r so that for $n \geq r$, $F(I)_n = (a_1^0, a_2^0, \dots, a_s^0)_n$. Thus $J I^{r-1} + \mathfrak{m} I^r = I^r$. By Nakayama's Lemma, $J I^{r-1} = I^r$. Hence J is a reduction of I . By Dimension Theorem $\mu(J) \geq \ell(I)$. \square

Proposition 3.7. *Let $J \subseteq I$ be a reduction of an ideal I in a local ring (R, \mathfrak{m}) . Then J contains a minimal reduction of I . Let $a_1, a_2, \dots, a_s \in J$ be such that $a_1^0, a_2^0, \dots, a_s^0 \in I/I\mathfrak{m}$ are linearly independent and s is minimal with respect to the property that $K = (a_1, a_2, \dots, a_s)$ is a reduction of I contained in J . Then K is a minimal reduction of I contained in J .*

Proof. Suppose $K' \subseteq K$ and K' be a reduction of I . Let $f : K/\mathfrak{m}K \rightarrow I/\mathfrak{m}I$ be the natural map of $k := R/\mathfrak{m}$ -vector spaces. Since $a_1^0, a_2^0, \dots, a_s^0 \in I/\mathfrak{m}I$ are k -linearly independent, $a_1 + \mathfrak{m}K, \dots, a_s + \mathfrak{m}K$ are k -linearly independent in $K/\mathfrak{m}K$. Hence $\text{Ker } f = K \cap \mathfrak{m}I/\mathfrak{m}K = 0$. Therefore $K \cap \mathfrak{m}I = \mathfrak{m}K$.

Next observe that $K' + \mathfrak{m}I = K + \mathfrak{m}I$. Indeed let $K' + \mathfrak{m}I < K + \mathfrak{m}I$. Then $K' + \mathfrak{m}I/\mathfrak{m}I$ is a proper subspace of $K + \mathfrak{m}I/\mathfrak{m}I$. Let $t = \dim(K' + \mathfrak{m}I/\mathfrak{m}I)$ and $b_1, b_2, \dots, b_t \in K'$ such that $b_1^0, b_2^0, \dots, b_t^0 \in I/I\mathfrak{m}$ are linearly independent. Since K' is a reduction of I , $\dim F(I)/(b_1^0, b_2^0, \dots, b_t^0) = 0$. This contradicts the minimality of s .

Thus $K \subseteq (K' + \mathfrak{m}I) \cap K = K' + (\mathfrak{m}I \cap K) = K' + \mathfrak{m}K$. By Nakayama's Lemma $K' = K$. Therefore K is a minimal reduction of I . \square

Proposition 3.8. *Let (R, \mathfrak{m}) be a local ring with infinite residue field k . Let I be an ideal of R and $a_1, a_2, \dots, a_s \in I$. Then $J = (a_1, a_2, \dots, a_s)$ is a minimal reduction of I if and only if $(a_1^0, a_2^0, \dots, a_s^0)$ is a homogeneous system of parameters of $F(I)$.*

Proof. Let J be a minimal reduction of I . Put $\ell = \ell(I)$. Then s is smallest with respect to the property that $\dim F(I)/(a_1^0, a_2^0, \dots, a_s^0) = 0$. Let $s > \ell(I)$. Since k is infinite, by Noether Normalization, there exist $b_1, b_2, \dots, b_\ell \in I$ such that $F(I)$ is integral over the polynomial ring $k[b_1^0, b_2^0, \dots, b_\ell^0]$. Hence $(b_1^0, b_2^0, \dots, b_\ell^0)F(I)$ is zero-dimensional. Therefore $(b_1, b_2, \dots, b_\ell)$ is a reduction of I . This contradicts minimality of s . Hence $s = \ell(I)$.

Conversely let $a_1, a_2, \dots, a_s \in I$ such that a_1^0, \dots, a_s^0 is a homogeneous system of parameters of $F(I)$. Then $(a_1, a_2, \dots, a_s) = J$ is a reduction of I . By the above proposition it is a minimal reduction of I . \square

Recall that *altitude* of an ideal I , $\text{alt } I$, is the maximum of the heights of the minimal primes over I .

Corollary 3.9. *Let I be an ideal of a local ring (R, \mathfrak{m}) . Then*

$$\text{alt } I \leq \ell(I) \leq \dim R.$$

Proof. We may assume that R/\mathfrak{m} is infinite. Let J be a minimal reduction of I . Then $V(I) = V(J)$. If p is any minimal prime of J . Then $\text{ht } p \leq \mu(J) = \ell(I)$. Since $\lambda(I^n/\mathfrak{m}I^n) \leq \lambda(I^n/I^{n+1})$ for all n , the degree of Hilbert polynomial of $F(I)$ is at most that of the Hilbert polynomial of $G(I)$. Hence $\ell(I) = \dim F(I) \leq \dim G(I) = \dim R$. \square

4. Hilbert function in 1-dimensional Cohen-Macaulay local rings

Throughout this section (A, \mathfrak{m}) is a 1-dimensional Cohen-Macaulay local ring with A/\mathfrak{m} infinite. Let I be an \mathfrak{m} -primary ideal. The Hilbert function of I is the function $H_I(n) = \lambda(A/I^n)$. The Hilbert polynomial of I , $P_I(n)$ has degree 1. Write

$$P_I(n) = e_0 n - e_1.$$

The *postulation number* of I is defined to be

$$n(I) = \max\{n \mid H_I(n) \neq P_I(n)\}.$$

Since $\text{ht } I = 1 = \dim A$, $\ell(I) = 1$. Since A/\mathfrak{m} is infinite, there exists $a \in I$ such that (a) is a reduction of I .

Theorem 4.1 (Northcott, 1960). *Let I be an \mathfrak{m} -primary ideal of a one dimensional Cohen-Macaulay local ring (A, \mathfrak{m}) . Let (a) be a minimal reduction of I . Then*

- (1) $P_I(n+1) - H_I(n+1) \geq P_I(n) - H_I(n)$ for all $n \geq 0$.
- (2) $e_0 - e_1 \leq \lambda(A/I)$.
- (3) $e_1 \geq 0$ with equality if and only if I is principal.

Proof. Notice that for all $n \geq 0$,

$$\begin{aligned} P_I(n+1) - H_I(n+1) &= (n+1)e_0 - e_1 - \lambda(A/I^{n+1}) \\ &= ne_0 - e_1 - \lambda((a)/aI^n) + \lambda(I^{n+1}/aI^n) \\ (3) \quad &= P_I(n) - H_I(n) + \lambda(I^{n+1}/aI^n). \end{aligned}$$

Hence $H_I(n) - P_I(n) \geq H_I(n+1) - P_I(n+1)$. For large n , $H_I(n) = P_I(n)$, hence for all $n \geq 0$, $H_I(n) \geq P_I(n)$. For $n = 1$ we get $\lambda(A/I) \geq e_0 - e_1$. Thus $e_1 \geq e_0 - \lambda(A/I) = \lambda(A/(a)) - \lambda(A/I) = \lambda(I/(a)) \geq 0$. Therefore if $e_1 = 0$ then $I = (a)$. Conversely if $I = (a)$ then for all $n \geq 1$,

$$\lambda(A/(a)^n) = \lambda(A/(a)) + \lambda((a)/(a)^2) + \cdots + \lambda((a)^{n-1}/(a)^n) = ne_0.$$

Hence $e_1(a) = 0$. \square

Proposition 4.2. *Let (A, \mathfrak{m}) be a one dimensional Cohen-Macaulay local ring. Let (a) be a minimal reduction of an \mathfrak{m} -primary ideal I . Then*

$$r_{(a)}I = n(I) + 1.$$

Proof. By (3), for all $n \geq 0$, we have

$$H_I(n+1) - P_I(n+1) = H_I(n) - P_I(n) - \lambda(I^{n+1}/aI^n).$$

Put $k = n(I)$ and $r = r_{(a)}I$. Hence for all $n \geq r$, $H_I(n) - P_I(n) = H_I(r) - P_I(r)$. But $H_I(n) = P_I(n)$ for large n . Hence $H_I(r) = P_I(r)$. Thus $k \leq r - 1$. To prove $k \geq r - 1$, put $n = k + 1$ in (3) to get

$$H_I(k+2) - P_I(k+2) = H_I(k+1) - P_I(k+1) - \lambda(I^{k+2}/aI^{k+1}).$$

Hence $\lambda(I^{k+2}/aI^{k+1}) = 0$. Thus $I^{k+2} = aI^{k+1}$ and hence $r \leq k + 1$. \square

Proposition 4.3. *Let (A, \mathfrak{m}) be a one dimensional Cohen-Macaulay local ring. Let (a) be a minimal reduction of an \mathfrak{m} -primary ideal I . Then $e_0 - e_1 = \lambda(A/I)$ if and only if $aI = I^2$.*

Proof. Let $aI = I^2$. Then $n(I) \leq 0$. Hence $H_I(1) = P_I(1)$. This gives $e_0 - e_1 = \lambda(A/I)$. Conversely let $e_0 - e_1 = \lambda(A/I)$. By (3), for $n \geq 1$,

$$0 \leq H_I(n) - P_I(n) \leq \lambda(A/I) - e_0 + e_1 = 0.$$

By Proposition 4.2, $n(I) = r_{(a)} - 1 \leq 0$. Hence $(a)I = I^2$. \square

Theorem 4.4 (Huckaba-Marley). *Let (A, \mathfrak{m}) be a one dimensional Cohen-Macaulay local ring. Let $J := (a)$ be a reduction of an \mathfrak{m} -primary ideal I . Then*

- (1) $e_1(I) = \sum_{n \geq 1} \lambda(I^n/JI^{n-1}) \geq \sum_{n \geq 1} \lambda(I^n/J \cap I^n)$.
- (2) $e_1(I) = \sum_{n \geq 1} \lambda(I^n/J \cap I^n)$ if and only if $G(I)$ is Cohen-Macaulay.

Proof. (Rossi) For all $m \geq 1$ we have

$$\lambda(I^{m-1}/I^m) = \lambda(A/J) - \lambda(I^m/JI^{m-1}) = e_0(I) - \lambda(I^m/JI^{m-1}).$$

Adding the above equation for $m = 1, 2, \dots, n$ we obtain

$$\lambda(A/I^n) = ne_0(I) - \sum_{m=1}^n \lambda(I^m/JI^{m-1}).$$

Taking n large in the above equation we get,

$$P_I(n) = ne_0(I) - e_1(I) = ne_0(I) - \sum_{m=1}^{r_J(I)} \lambda(I^m/JI^{m-1}).$$

Thus $e_1(I) = \sum_{m=1}^{r_J(I)} \lambda(I^m/JI^{m-1})$. Hence $e_1(I) \geq \sum_{m=1}^{r_J(I)} \lambda(I^m/J \cap I^m)$ and equality holds if and only if $I^m \cap J = JI^{m-1}$ for all $m \geq 1$. The last condition is equivalent to $G(I)$ being Cohen-Macaulay. \square

Example 4.5. Let k be a field and t be an indeterminate. Let $A = k[[t^3, t^4, t^5]]$ and $\mathfrak{m} = (t^3, t^4, t^5)$. Check that $t^3\mathfrak{m} = \mathfrak{m}^2$. Since A is Cohen-Macaulay, $e_0(\mathfrak{m}) = e_0(t^3) = \lambda(A/t^3A) = 3$. Since $t^3\mathfrak{m} = \mathfrak{m}^2$ and $e_0 - e_1 = \lambda(A/\mathfrak{m}) = 1$, we get $e_1(\mathfrak{m}) = 2$. Therefore $P_{\mathfrak{m}}(n) = 3n - 2$ for all $n \geq 1$.

5. Superficial sequences

In this section we develop the theory of superficial elements and superficial sequences. Existence of a superficial element in an ideal I allows us to relate Hilbert coefficients of I and those of $I/(a)$. As a result we can first investigate Hilbert coefficients in dimension one and lift this information to higher dimension.

Definition 5.1. *Let I be an ideal of a local ring (A, \mathfrak{m}) . We say $a \in I$ is superficial if there is a $c \geq 0$ such that $(I^n : a) \cap I^c = I^{n-1}$ for all $n > c$.*

Proposition 5.2. *Let I be an ideal of a local ring (A, \mathfrak{m}) .*

- (1) *If I is nilpotent then every $a \in I$ is superficial for I .*
- (2) *If I is not nilpotent then a superficial element a of I satisfies $a \in I \setminus I^2$.*

Proof. (1) Let $I^r = 0$. Then for $c = r + 1$ and $n > r + 1$, we have $(I^n : a) \cap I^c = I^{n-1} = 0$ for any $a \in I$. Hence a is superficial for I .

(2) Suppose I is not nilpotent. Suppose a is superficial for I and for all $n > c$, $(I^n : a) \cap I^c = I^{n-1}$. Suppose $a \in I^2$. Put $n = c + 2$ to get $(I^{c+2} : a) \cap I^c = I^{c+1}$. As $aI^c \subseteq I^{c+2}$, $I^c = I^{c+1}$. By Nakayama's Lemma $I^c = 0$. This is a contradiction. Hence $a \in I \setminus I^2$. \square

Proposition 5.3. *Let (A, \mathfrak{m}) be a local ring and let I be an ideal. Let $a \in I \setminus I^2$ and $a^* = a + I^2$. Then a is superficial for I if and only if the multiplication map $a^* : I^n/I^{n+1} \rightarrow I^{n+1}/I^{n+2}$ is injective for large n .*

Proof. Let $(I^n : a) \cap I^c = I^{n-1}$ for all $n > c$. Suppose $n > c$, $b \in I^n$ and $b^*a^* = 0$. Then $ba \in I^{n+2}$. Therefore $b \in (I^{n+2} : a) \cap I^c = I^{n+1}$. So $b^* = 0$. Hence the map a^* is injective for large n .

Conversely let $a^* : I^n/I^{n+1} \rightarrow I^{n+1}/I^{n+2}$ be injective for $n > c$. We show $(I^n : a) \cap I^c = I^{n-1}$ for all $n > c$. Let $b \in (I^n : a) \cap I^c$. Let $b \in I^m \setminus I^{m+1}$. Then $b^*a^* \in I^{m+1}/I^{m+2}$. If $b^*a^* = 0$ then $ab \in I^{m+2}$. Thus $b \in (I^{m+2} : a) \cap I^c = I^{m+1}$ which is a contradiction. Therefore $b^*a^* \neq 0$ and consequently $ab \notin I^{m+2}$ and $ab \in I^n$. Thus $n < m + 2$. Hence $b \in I^m \subseteq I^{n-1}$ and $(I^n : a) \cap I^c = I^{n-1}$. \square

Existence of superficial elements

Proposition 5.4. *Let (A, \mathfrak{m}) be a local ring with A/\mathfrak{m} infinite. Let M be an A -module. If N_1, N_2, \dots, N_t are proper submodules of M then*

$$N_1 \cup N_2 \cup \dots \cup N_t < M.$$

Proof. We apply induction on t . The $t = 1, 2$ cases are trivial. Suppose that $t \geq 3$ and let $M = N_1 \cup N_2 \cup \dots \cup N_t$. We may assume that $N_1 \not\subseteq N_2 \cup N_3 \cup \dots \cup N_t$ and $N_2 \cup N_3 \cup \dots \cup N_t \not\subseteq N_1$. As A/\mathfrak{m} is infinite there are units $u_1, u_2, \dots \in A$ such that $u_i - u_j$ is a unit for $i \neq j$ in A . Let

$a \in N_1 \setminus (N_2 \cup N_3 \cup \dots \cup N_t)$ and $b \in (N_2 \cup N_3 \cup \dots \cup N_t) \setminus N_1$. Then $a + ub$ and $a + wb \in N_j$ for some j and distinct units $u, w \in A$ by Pigeon-Hole Principle. Since

$$(a + ub) - (a + wb) = (u - w)b \in N_j,$$

and $u - w$ is a unit, $b \in N_j$. By choice of b , $j \neq 1$. Since $w - u$ is a unit and

$$w(a + ub) - u(a + wb) = (w - u)a \in N_j,$$

we conclude $a \in N_j$. The choice of a forces $j = 1$. This is a contradiction. Thus $N_1 \cup N_2 \cup \dots \cup N_t < M$. \square

Theorem 5.5. *Let (A, \mathfrak{m}) be a local ring with A/\mathfrak{m} infinite. Let I, J_1, \dots, J_t be A -ideals with $I \not\subseteq J_1 \cup \dots \cup J_t$. Then there exists $a \in I \setminus (J_1 \cup \dots \cup J_t)$ such that a is superficial for I .*

Proof. First note that the A/I -submodules $(J_i \cap I) + I^2/I^2$, $i = 1, 2, \dots, t$ are proper submodules of I/I^2 . Indeed, let $(J_i \cap I) + I^2 = I$. By Nakayama's Lemma $J_i \cap I = I$. Hence $I \subseteq J_i$, which is a contradiction. Let

$$(0) = Q_1 \cap \dots \cap Q_s \cap Q_{s+1} \cap \dots \cap Q_g$$

be a reduced primary decomposition of (0) in $G(I)$. Put $G_n = I^n/I^{n+1}$. Let $\sqrt{Q_i} = P_i$ for $i = 1, 2, \dots, g$. Suppose $G_1 \not\subseteq P_i$ for $i = 1, \dots, s$ and $G_1 \subseteq P_j$ for $j = s+1, \dots, g$. Therefore $G_1 \cap P_1, \dots, G_1 \cap P_s$ are proper G_0 -submodules of G_1 . By previous proposition, there is an $a \in I \setminus I^2$ such that

$$a^* \in G_1 \setminus \{[\cup_{i=1}^s P_i] \cup [\cup_{i=1}^t (J_i \cap I) + I^2/I^2]\}.$$

We claim that a is superficial for I . For this it is enough to show that $(0 : a^*) \cap G_n = 0$ for large n . Suppose $b^* a^* = 0$. Since $a^* \notin P_i$ for $i = 1, \dots, s$, $b^* \in (Q_1 \cap \dots \cap Q_s)$. Since Q_j is P_j -primary for all j , $P_j^N \subseteq Q_j$ for N large. Hence $G_1^N = I^N/I^{N+1} \subseteq Q_j$ for $j = s+1, \dots, g$. Therefore

$$G_N \cap (0 : a^*) \subseteq Q_1 \cap \dots \cap Q_s \cap Q_{s+1} \cap \dots \cap Q_g = (0).$$

Hence a is superficial for I . \square

Superficial sequences and reductions

Definition 5.6. *Let (A, \mathfrak{m}) be a local ring, and let I be an A -ideal. A sequence $x_1, x_2, \dots, x_s \in I$ is called a superficial sequence for I or I -superficial sequence if $\overline{x_i}$ is superficial for $I/(x_1, \dots, x_{i-1})$ for $i = 1, 2, \dots, s$.*

Lemma 5.7. *Let x_1, \dots, x_s be an I -superficial sequence for I . Then for $n \gg 0$,*

$$I^n \cap (x_1, \dots, x_s) = (x_1, \dots, x_s)I^{n-1}.$$

Proof. Induct on s . Let $s = 1$. As x_1 is superficial for I , there is $c \geq 0$ such that for $n > c$,

$$(I^{n+1} : x_1) \cap I^c = I^n.$$

By Artin-Rees Lemma, there is a p such that $I^n \cap x_1 A = I^{n-p}(I^p \cap x_1 A) \subseteq x_1 I^{n-p}$ for all $n \geq p$. We now show $I^n \cap x_1 A = x_1 I^{n-1}$ for all $n \geq c + p$. Let $y = bx_1 \in I^n \cap x_1 A$ for some $b \in A$. Then $y \in I^{n-p}x_1$. Hence $y = dx_1$, where $d \in I^{n-p} \subseteq I^c$, since $n - p \geq c$. Therefore $(b - d)x_1 = 0$. Hence $b - d \in (0 : x_1) \subseteq (I^n : x_1)$. Now $d = b - (b - d) \in (I^n : x_1) \cap I^c = I^{n-1}$. Hence $I^n \cap x_1 A = x_1 I^{n-1}$ for $n \geq p + c$. Now let $s \geq 2$. By induction, for large n ,

$$I^n \cap (x_1, \dots, x_{s-1}) = (x_1, \dots, x_{s-1})I^{n-1}.$$

As \bar{x}_s is superficial for $A/(x_1, \dots, x_{s-1})$, by $s = 1$ case, for large n ,

$$(\bar{x}_s) \cap [I/(x_1, \dots, x_{s-1})]^n = (\bar{x}_s)[I/(x_1, \dots, x_{s-1})]^{n-1}.$$

Hence for large n ,

$$\begin{aligned} x_s I^{n-1} + (x_1, \dots, x_{s-1}) &= (x_1, \dots, x_s) \cap [I^n + (x_1, \dots, x_{s-1})] \\ &= I^n \cap (x_1, \dots, x_s) + (x_1, \dots, x_{s-1}). \end{aligned}$$

Therefore

$$\begin{aligned} I^n \cap (x_1, \dots, x_s) &\subseteq x_s I^{n-1} + (x_1, \dots, x_{s-1}) \cap I^n \\ &= x_s I^{n-1} + (x_1, \dots, x_{s-1}) I^{n-1} \\ &= (x_1, \dots, x_{s-1}) I^{n-1}. \end{aligned}$$

□

Lemma 5.8. *Let $J \subseteq I$ be ideals of a local ring A . . Let $a \in J$ be superficial for I . If $J/(a)$ is a reduction of $I/(a)$ then J is a reduction of I .*

Proof. Let $J/(a)$ be a reduction of $I/(a)$. Then for large n , $J I^n + (a) = I^{n+1} + (a)$. Hence $I^{n+1} \subseteq J I^n + (a) \cap I^{n+1} = J I^n + a I^n = J I^n$ for large n . Hence $J I^n = I^{n+1}$. Thus J is a reduction of I . □

Proposition 5.9. *Let I be an ideal of a local ring (A, \mathfrak{m}) . Let a be superficial for I and $a^* = a + I^2$. Then for large n ,*

$$[G(I)/a^*]_n \cong G(I/aA)_n.$$

Proof. On one hand we have for large n ,

$$G(I/aA)_n = \frac{I^n + aA}{I^{n+1} + aA} \cong \frac{I^n}{I^{n+1} + (aA \cap I^n)} = \frac{I^n}{I^{n+1} + aI^{n-1}}.$$

On the other hand for large n ,

$$[G(I)/a^*]_n = \frac{I^n/I^{n+1}}{aI^{n-1} + I^{n+1}/I^{n+1}} \cong \frac{I^n}{aI^{n-1} + I^{n+1}}.$$

□

Corollary 5.10. *Let (A, \mathfrak{m}) be a d -dimensional local ring and let I be an \mathfrak{m} -primary ideal. Suppose a is superficial for I . Then (a) is parameter, i.e. $\dim A/aA = d - 1$. Moreover if $d = 1$ then (a) is a reduction of I .*

Proof. By Lemma 5.7, $I^n \cap aA = aI^{n-1}$ for large n . Consider the exact sequence

$$0 \longrightarrow (0 : a^*)_n \longrightarrow G(I)_n \xrightarrow{a^*} G(I)_n \longrightarrow [G(I)/a^*]_n \longrightarrow 0.$$

As a is superficial for I , $(0 : a^*)_n = 0$ for large n . Hence for large n ,

$$\lambda(I^n/I^{n+1}) - \lambda(I^{n-1}/I^n) = \lambda[G(I)/a]_n.$$

Thus $\lambda[G(I)/a]_n$ is a polynomial function of degree $d-2$. Hence $\dim A/aA = d - 1$.

If $d = 1$, then A/aA is Artin. Hence aA is \mathfrak{m} -primary. Therefore $I^n \subseteq aA$ for large n . But $I^n \cap aA = aI^{n-1}$ for large n . Hence for large n , $I^n = aI^{n-1}$. Thus (a) is a reduction of I . \square

Theorem 5.11. *Let (A, \mathfrak{m}) be a d -dimensional local ring. Let a_1, \dots, a_d be a superficial sequence for I . Then $J = (a_1, \dots, a_d)$ is a minimal reduction of I .*

Proof. Apply induction on d . We have proved this for $d = 1$. Since $\overline{a_2}, \overline{a_3}, \dots, \overline{a_d}$ is a superficial sequence for $I/(a_1)$ in the $(d-1)$ -dimensional local ring A/a_1A , $\overline{a_2}, \dots, \overline{a_d}$ is a reduction of $I/(a_1)$. By Lemma 5.8, J is a reduction of I . \square

Proposition 5.12. *Let (A, \mathfrak{m}) be a d -dimensional local ring with A/\mathfrak{m} is infinite. Let J be a minimal reduction of an \mathfrak{m} -primary ideal I . Then J can be generated by a superficial sequence for I .*

Proof. Put $J = (b_1, b_2, \dots, b_d)$ and $b_i^* = b + I^2$, for $i = 1, 2, \dots, d$. Then $(b_1^*, \dots, b_d^*)_n = (JI^{n-1} + I^{n+1})/I^{n+1}$. Hence

$$[G(I)/(b_1^*, \dots, b_d^*)_n] \cong I^n/(JI^{n-1} + I^{n+1}) = 0 \text{ for } n \geq r_J(I) + 1.$$

Thus b_1^*, \dots, b_d^* is a homogeneous system of parameters for $G(I)$. Apply induction on d . If $d = 0$ then (0) is a reduction of I . Let $d \geq 1$ and P_1, \dots, P_s be the relevant associated primes of $G(I)$. In other words $G_1 \not\subseteq P_i$ for $i = 1, 2, \dots, s$. Then $\dim G(I) = d > \text{ht } P_i$ for all i . Moreover $J + I^2/I^2 \not\subseteq P_i$ for all i . By the prime avoidance lemma for homogeneous ideals, there is an $a_1 \in J \setminus \mathfrak{m}J$ such that $a_1^* \notin \cup_{i=1}^s P_i$. Hence a_1 is superficial for I . Since $J/(a_1)$ is a minimal reduction of $I/(a_1)$, by induction, there exists a superficial sequence $\overline{a_2}, \dots, \overline{a_{d-1}}$ for $I/(a_1)$ such that $J/(a_1) = (\overline{a_2}, \dots, \overline{a_{d-1}})$. Thus $J = (a_1, \dots, a_d)$. \square

Superficial elements and Hilbert polynomials

Let I be an \mathfrak{m} -primary ideal of a d -dimensional local ring (A, \mathfrak{m}) with A/\mathfrak{m} infinite. The Hilbert polynomial of I , $P_I(n)$, is written as

$$P_I(n) = e_0(I) \binom{n+d-1}{d} - e_1(I) \binom{n+d-2}{d-1} + \cdots + (-1)^d e_d(I).$$

We have seen that if a is superficial for I then $\dim A/aA = d-1$. We will now study the relationship between Hilbert coefficients of I and those of I/aA . We will see that superficial sequences provide us with an inductive tool to study the Hilbert polynomial. Let $f : \mathbb{N} \rightarrow \mathbb{N}$ be a function. Put $\Delta f(n) = f(n) - f(n-1)$.

Theorem 5.13. *Let I be an \mathfrak{m} -primary ideal of a d -dimensional local ring (A, \mathfrak{m}) . Let a be a superficial element for I . Let $\bar{A} = A/(a)$ and $\bar{I} = I/(a)$. Then*

- (1) $P_{\bar{I}}(n) = \Delta P_I(n) + \lambda(0 : a)$. Hence $\dim A/(a) = d-1$.
- (2) For $i = 0, 1, \dots, d-2$, $e_i(\bar{I}) = e_i(I)$ and $e_{d-1}(\bar{I}) = e_{d-1}(I) + \lambda(0 : a)$.

Proof. By the exact sequence

$$0 \longrightarrow (I^n : a)/I^{n-1} \longrightarrow A/I^{n-1} \xrightarrow{a} A/I^n \longrightarrow A/(I^n, a) \longrightarrow 0$$

we get $\lambda(A/(I^n, a)) = \lambda(A/I^n) - \lambda(A/I^{n-1}) + \lambda((I^n : a)/I^{n-1})$. Thus for large n ,

$$P_{\bar{I}}(n) = \Delta P_I(n) + \lambda((I^n : a)/I^{n-1}).$$

Since a is superficial for I , there exists $c \geq 0$ such that $(I^n : a) \cap I^c = I^{n-1}$ for all $n > c$. Hence the map $\varphi : (I^n : a)/I^{n-1} \rightarrow A/I^c$, $\varphi(\bar{b}) = b + I^c$ has kernel $(I^n : a) \cap I^c/I^{n-1}$ which is 0 for large n . Hence for large n , $\lambda((I^n : a)/I^{n-1}) \leq \lambda(A/I^c)$. We proceed to prove that for large n ,

$$\lambda((I^n : a)/I^{n-1}) = \lambda(0 : a).$$

From the exact sequence

$$0 \longrightarrow A/I^{n-1} \longrightarrow A/I^c \oplus A/(I^n : a) \longrightarrow A/(I^c + (I^n : a)) \longrightarrow 0$$

we get

$$\lambda(A/I^c) + \lambda(A/(I^n : a)) = \lambda(A/I^{n-1}) + \lambda[A/(I^c + (I^n : a))].$$

Hence $\lambda((I^n : a)/I^{n-1}) = \lambda(I^c + (I^n : a)/I^c)$. Notice that $I^c + (I^n : a) = (0 : a) + I^c$ for all $n > c$. Indeed, let $b \in (I^n : a)$. Then for large $n \geq p$, by Artin-Rees Lemma, we get,

$$ba \in (I^n : a)a = I^n \cap (a) = I^{n-p}(I^p \cap (a)) \subseteq aI^{n-p}.$$

Hence $ba = ay$ for some $y \in I^{n-p}$. Thus $b - y \in (0 : a)$ and so $b \in (0 : a) + I^{n-p} \subseteq (0 : a) + I^c$ for large n . Hence

$$\lambda((I^n : a)/I^{n-1}) = \lambda((0 : a) + I^c)/I^c = \lambda((0 : a)/(I^c \cap (0 : a))).$$

But $I^c \cap (0 : a) \subseteq I^c \cap (I^n : a) = I^{n-1}$ for all large n . By Krull Intersection Theorem, $I^c \cap (0 : a) = 0$. Hence

$$P_{\bar{I}}(n) = \Delta P_I(n) + \lambda(0 : a).$$

Since $\dim A/(a) = \deg P_{\bar{I}}(n) = \deg P_I(n) - 1 = d - 1$, a is a parameter for A . The equation above gives $e_0(\bar{I}) = e_0(I) + \lambda(0 : a)$ when $d = 1$ and for $d \geq 2$, $e_i(\bar{I}) = e_i(I)$ for $i = 0, 1, \dots, d - 2$ and $e_{d-1}(\bar{I}) = e_{d-1}(I) + \lambda(0 : a)$. \square

Theorem 5.14 (Sally-Machine). *Let (A, \mathfrak{m}) be a local ring. Let x be superficial for I . Suppose $\text{depth } G(I/(x)) > 0$. Then x^* is $G(I)$ -regular.*

Proof. (B. Singh) Put $G(I) = \bigoplus_{n=0}^{\infty} G_n$ where $G_n = I^n/I^{n+1}$. We will show that $(x^*)^s$ is $G(I)$ -regular for all s . We will use induction on n to show that for all $s \geq 0$, $G_n \cap (0 : (x^*)^s) = 0$. Let $f : G(I) \rightarrow G(I/(x))$ be the natural map. Then $f(G_0) = G(I/(x))_0 = G_0$.

Note that $f(0 : (x^*)^s) = 0$. Indeed, since x is I -superficial, $(x^*)^s : G_n \rightarrow G_{n+1}$ is injective for large n and all $s \geq 1$. Hence $(0 : (x^*)^s)G_n \subseteq (0 : (x^*)^s) \cap G_n = 0$ for large n . Therefore $f(G_n)f(0 : (x^*)^s) = 0$ for large n . But $f(G_n) = G(I/xA)_n$ has a $G(I)$ -regular element for large n as $\text{depth } G(I/xA) > 0$. Thus $f(0 : (x^*)^s) = 0$.

Let $\bar{a} \in G_0 \cap (0 : (x^*)^s)$. Then $f(\bar{a}) = \bar{a} \in f((0 : (x^*)^s)) = 0$. Suppose that for all $r = 0, 1, \dots, n - 1$, $G_r \cap (0 : (x^*)^s) = 0$. Let $b \in I^n \setminus I^{n+1}$ and $b^* \in G_n \cap (0 : (x^*)^s)$. Then $b^*(x^*)^s = 0$. Hence $bx^s \in I^{n+s+1}$. Since $f(b^*) = 0$, $b \in I^{n+1} + xA$. Let $b = c + dx$ for $c \in I^{n+1}$, $d \in A$. If $d \in I^n$ then $b \in I^{n+1}$ which is a contradiction. So let $d \in I^t \setminus I^{t+1}$, where $t < n$. Since $cx^s = bx^s - dx^{s+1} \in I^{n+s+1}$, we get $dx^{s+1} \in I^{n+s+1} \subset I^{t+s+1}$. Hence $d^*(x^*)^{s+1} = 0$. Thus $d^* \in G_t \cap (0 : (x^*)^{s+1}) = 0$ by induction. This is a contradiction. Hence $d \in I^n$ and therefore $b \in I^{n+1}$, thus $b^* = 0$. \square

Proposition 5.15. *Let (A, \mathfrak{m}) be a local ring. Let I be \mathfrak{m} -primary. Let x_1, x_2, \dots, x_r be a superficial sequence for I . Suppose that $\text{depth } A \geq r$. Then x_1, x_2, \dots, x_r is an A -regular sequence.*

Proof. Apply induction on r . Let $r = 1$. Since x_1 is superficial, there is a c such that $(I^n : x_1) \cap I^c = I^{n-1}$ for all $n > c$. Thus $(0 : x_1) \cap I^c \subseteq I^{n-1}$ for all n large. By Krull Intersection Theorem, $(0 : x_1) \cap I^c = 0$. But $\text{depth } A = \text{grade } I > 0$, hence I^c has a regular element, say a . Then $(0 : x_1)a \subseteq (0 : x_1) \cap I^c = 0$. Therefore $(0 : x_1) = 0$.

As $\bar{x}_2, \bar{x}_3, \dots, \bar{x}_r$ is A/x_1A -superficial sequence and $\text{depth } A/x_1A \geq r - 1$, by induction $\bar{x}_2, \bar{x}_3, \dots, \bar{x}_r$ is an A/x_1A -regular sequence. Hence x_1, x_2, \dots, x_r is an A -regular sequence. \square

We end this section by proving an important criterion due to Valabrega and Valla [23] for a sequence of initial forms $x_1^*, x_2^*, \dots, x_s^*$ in I/I^2 to be a $G(I)$ -regular sequence.

Theorem 5.16 (Valabrega-Valla, 1978). *Let (R, \mathfrak{m}) be a local ring. Let I be an ideal of R . Let $x_1, x_2, \dots, x_s \in I \setminus I^2$. Then $x_1^*, x_2^*, \dots, x_s^*$ is a $G(I)$ -regular sequence if and only if x_1, x_2, \dots, x_s is an R -sequence and for all $n \geq 1$,*

$$(x_1, x_2, \dots, x_s) \cap I^n = (x_1, x_2, \dots, x_s)I^{n-1}.$$

Proof. Apply induction on s . Let $s = 1$ and put $x_1 = x$. Let x^* be $G(I)$ -regular. Let $a \in R$ and $ax = 0$. If $a \neq 0$, there is an m such that $a \in I^m \setminus I^{m+1}$. Then $a^*x^* = 0$. Hence $a^* = 0$, which is a contradiction. Thus x is R -regular. Next we show $(x) \cap I^n = xI^{n-1}$ for $n \geq 1$. Let $b \in I^m \setminus I^{m+1}$ and $bx \in I^n$. Then $b^*x^* \in I^{m+1}/I^{m+2}$. Since x^* is $G(I)$ -regular and $b^* \neq 0$, $b^*x^* \neq 0$. Thus $bx \notin I^{m+2}$. Therefore $n - 1 \leq m$ and so $b \in I^{n-1}$.

Conversely let x be R -regular and $(x) \cap I^n = xI^{n-1}$ for $n \geq 1$. Let $b^* \in I^m/I^{m+1}$ and $b^*x^* = 0$, then $bx \in I^{m+2} \cap xR = I^{m+1}x$. As x is regular in R , $b \in I^{m+1}$. Hence $b^* = 0$.

Now assume the result for $s - 1$, $s \geq 2$. Let $x_1^*, x_2^*, \dots, x_s^*$ be $G(I)$ -regular. Let $S = R/(x_1)$ and $J = I/(x_1)$. Let “ $-$ ” denote images in S . Since x_1^* is $G(I)$ -regular, $G(I/x_1) \simeq G(I)/(x_1^*)$. Hence $\overline{x_2^*}, \dots, \overline{x_s^*}$ is a $G(I/(x_1))$ -regular sequence. By induction hypothesis $\overline{x_2}, \dots, \overline{x_s}$ is $R/(x_1)$ -regular sequence and for $n \geq 1$,

$$(4) \quad J^n \cap (\overline{x_2}, \dots, \overline{x_s}) = (\overline{x_2}, \dots, \overline{x_s})J^{n-1}.$$

Since x_1^* is $G(I)$ -regular, x_1 is R -regular. Hence x_1, x_2, \dots, x_s is an R -regular sequence. We need to prove for $n \geq 1$,

$$(5) \quad I^n \cap (x_1, x_2, \dots, x_s) = (x_1, x_2, \dots, x_s)I^{n-1}.$$

Let $r_1x_1 + \dots + r_sx_s \in I^n$ for some $r_1, \dots, r_s \in R$. Then

$$\overline{r_2x_2} + \dots + \overline{r_sx_s} \in J^n \cap (\overline{x_2}, \dots, \overline{x_s}) = J^{n-1}(\overline{x_2}, \dots, \overline{x_s}).$$

Hence $\overline{r_2x_2} + \dots + \overline{r_sx_s} = \overline{t_2x_2} + \dots + \overline{t_sx_s}$ for some $t_1, \dots, t_s \in I^{n-1}$. Thus for some $t_1 \in R$,

$$(r_2 - t_2)x_2 + \dots + (r_s - t_s)x_s = t_1x_1$$

Hence

$$(r_1 + t_1)x_1 = (r_1x_1 + \dots + r_sx_s) - (t_2x_2 + \dots + t_sx_s) \in I^n.$$

Therefore $(r_1 + t_1) \in I^{n-1}$. This gives

$$r_1x_1 + \dots + r_sx_s \in (x_1, x_2, \dots, x_s)I^{n-1}.$$

Conversely let x_1, x_2, \dots, x_s be an R -sequence and let (5) hold for all $n \geq 1$. Suppose we prove

$$(6) \quad I^n \cap (x_1, x_2, \dots, x_{s-1}) = (x_1, x_2, \dots, x_{s-1})I^{n-1}$$

for all $n \geq 1$, then by $s-1$ case $x_1^*, x_2^*, \dots, x_{s-1}^*$ is a $G(I)$ -regular sequence. Thus

$$G(I)/(x_1^*, x_2^*, \dots, x_{s-1}^*) \simeq G(I)/(x_1, x_2, \dots, x_{s-1}).$$

By $s=1$ case, x_s^* is $G(I)/(x_1^*, x_2^*, \dots, x_{s-1}^*)$ -regular. Hence $x_1^*, x_2^*, \dots, x_s^*$ is $G(I)$ -regular.

We prove (6) by induction on n . The $n=1$ case is clear. Let $n \geq 2$ and $r_1 s_1 + \dots + r_{s-1} x_{s-1} \in I^n$ for some $r_1, r_2, \dots, r_{s-1} \in R$. Then

$$r_1 s_1 + \dots + r_{s-1} x_{s-1} \in I^n \cap (x_1, x_2, \dots, x_s) = (x_1, x_2, \dots, x_s) I^{n-1}.$$

Hence there exist $t_1, \dots, t_s \in I^{n-1}$ such that

$$r_1 x_1 + \dots + r_{s-1} x_{s-1} = x_1 t_1 + \dots + x_s t_s.$$

Hence $t_s x_s \in (x_1, x_2, \dots, x_{s-1})$. As x_1, x_2, \dots, x_s is an R -sequence,

$$t_s \in (x_1, x_2, \dots, x_{s-1}) \cap I^{n-1} = (x_1, x_2, \dots, x_{s-1}) I^{n-2}.$$

Therefore $t_s x_s \in I^{n-1}(x_1, x_2, \dots, x_{s-1})$. Hence

$$r_1 x_1 + \dots + r_{s-1} x_{s-1} \in I^{n-1}(x_1, x_2, \dots, x_{s-1}).$$

□

6. Huckaba-Marley Theorem

In this section we prove, by classical techniques, theorems due to Huckaba and Marley which characterize Cohen-Macaulay property of $G(I)$ and $\text{depth } G(I) \geq \dim R - 1$, in terms of $e_1(I)$. For another proof of these theorems using Rees algebras, we refer the reader to [11]. We first prove the results in dimension one and then using the Sally machine, we are able to prove them in general. We will also prove several consequences of these theorems.

Theorem 6.1 (Huckaba-Marley; 1996, 1997). *Suppose (R, \mathfrak{m}) is a Cohen-Macaulay local ring of dimension d with $k = R/\mathfrak{m}$ infinite. Let J be a minimal reduction of I . Then*

- (1) $\sum_{n \geq 1} \lambda(I^n/J \cap I^n) \leq e_1(I) \leq \sum_{n \geq 1} \lambda(I^n/JI^{n-1})$.
- (2) $e_1(I) = \sum_{n \geq 1} \lambda(I^n/J \cap I^n)$ if and only if $G(I)$ is Cohen-Macaulay.
- (3) $e_1(I) = \sum_{n \geq 1} \lambda(I^n/JI^{n-1})$ if and only if $\text{depth } G(I) \geq d-1$.

Proof. Apply induction on d . The $d=1$ case is already proved in Theorem 4.4.

Now let $d \geq 2$. Assume the theorem for $d - 1$. Let $J = (a_1, \dots, a_d)$ where a_1, \dots, a_d is a superficial sequence for I . Let “ $\bar{\cdot}$ ” denote images in $\bar{R} = R/a_1R$. By induction hypothesis,

$$\begin{aligned} e_1(I) = e_1(I/a_1R) &\leq \sum_{i=1}^{\infty} \lambda(\bar{I}^i / \bar{J}I^{i-1}) = \sum_{i=1}^{\infty} \lambda\left(\frac{I^i + a_1R}{JI^{i-1} + a_1R}\right) \\ &= \sum_{i=1}^{\infty} \lambda\left(\frac{I^i}{JI^{i-1} + I^i \cap a_1R}\right) \\ &\leq \sum_{i=1}^{\infty} \lambda(I^i / JI^{i-1}). \end{aligned}$$

By induction hypothesis we also have

$$e_1(I) = e_1(I/a_1R) \geq \sum_{i=1}^{\infty} \lambda\left(\frac{\bar{I}^i + \bar{J}}{\bar{J}}\right) = \sum_{i=1}^{\infty} \lambda\left(\frac{I^i + J}{J}\right).$$

(2) Now we show that $G(I)$ is Cohen-Macaulay if and only if $e_1(I) = \sum_{i=1}^{\infty} \lambda(I^i + J/J)$. Let $G(I)$ be Cohen-Macaulay. Let $J = (a_1, \dots, a_d)$ be a minimal reduction of I . Then a_1^*, \dots, a_d^* in I/I^2 is a $G(I)$ -regular sequence. Thus $G(I/a, R) \cong G(I)/(a^*)$ is Cohen-Macaulay. By induction

$$e_1(I) = e_1(I/a_1R) = \sum_{n=1}^{\infty} \lambda\left(\frac{\bar{I}^n + \bar{J}}{\bar{J}}\right) = \sum_{n=1}^{\infty} \lambda\left(\frac{I^n + J}{J}\right).$$

Conversely let $e_1(I) = \sum_{n=1}^{\infty} \lambda(I^n + J/J)$. Then $e_1(I/a_1R) = e_1(I) = \sum_{n=1}^{\infty} \lambda(\bar{I}^n + \bar{J}/\bar{J})$. By induction hypothesis $G(I/a, R)$ is Cohen-Macaulay. By Sally-machine a_1^* is $G(I)$ -regular. Therefore $G(I/a_1, R) = G(I)/(a_1^*)$. Hence $\bar{a}_2^*, \dots, \bar{a}_d^*$ in $G(I)/(a_1^*)$ is a regular sequence. Therefore $G(I)$ is Cohen-Macaulay.

(3) Let $e_1(I) = \sum_{n=1}^{\infty} \lambda(I^n / JI^{n-1})$. Let a_1, \dots, a_d be an I -superficial sequence generating J . Let $K = (a_1, \dots, a_{d-1})$. Then

$$\begin{aligned} e_1(I) = e_1(I/K) &= \sum_{n=1}^{\infty} \lambda\left(\frac{I^n + K}{a_d I^{n-1} + K}\right) \\ &= \sum_{n=1}^{\infty} \lambda\left(\frac{I^n}{a_d I^{n-1} + K \cap I^n}\right) \\ &= \sum_{n=1}^{\infty} \lambda(I^n / JI^{n-1}). \end{aligned}$$

Therefore $a_d I^{n-1} + K \cap I^n = JI^{n-1}$ for all $n \geq 1$. Thus $K \cap I^n \subseteq JI^{n-1}$ for all $n \geq 1$. By the next lemma a_1^*, \dots, a_{d-1}^* is $G(I)$ -regular. Conversely let $\text{depth } G(I) \geq d - 1$. Let $J = (a_1, \dots, a_d)$ be a minimal reduction of I

such that $a_1^* \in I/I^2$ is $G(I)$ -regular. Then

$$\begin{aligned}
 e_1(I) = e_1(I/a_1R) &= \sum_{n=1}^{\infty} \lambda(\overline{I}^n / \overline{JI}^{n-1}) \\
 &= \sum_{n=1}^{\infty} \lambda\left(\frac{I^n + a_1R}{JI^{n-1} + a_1R}\right) \\
 &= \sum_{n=1}^{\infty} \lambda\left(\frac{I^n}{JI^{n-1} + a_1R \cap I^n}\right) \\
 &= \sum_{n=1}^{\infty} \lambda\left(\frac{I^n}{JI^{n-1} + a_1I^{n-1}}\right) \\
 &= \sum_{n=1}^{\infty} \lambda(I^n / JI^{n-1}).
 \end{aligned}$$

□

Lemma 6.2. *Let (R, \mathfrak{m}) be a d -dimensional Cohen-Macaulay local ring and let I be an \mathfrak{m} -primary ideal. Suppose $(x_1, x_2, \dots, x_d) = J$ is a minimal reduction of I such that*

$$I^n \cap (x_1, \dots, x_{d-1}) \subseteq JI^{n-1}$$

for all $n \geq 1$. Then x_1^*, \dots, x_{d-1}^* is a $G(I)$ -regular sequence.

Proof. By Valabrega-Valla Theorem it is enough to prove that

$$(7) \quad I^n \cap (x_1, \dots, x_{d-1}) = (x_1, \dots, x_{d-1})I^{n-1}$$

for all $n \geq 1$. For $n = 1$, (7) is clearly true. Suppose (7) is true for $n - 1$. To prove (7) for n , let $z \in I^n \cap (x_1, \dots, x_{d-1})$. Then

$$z = r_1x_1 + \dots + r_{d-1}x_{d-1} = s_1x_1 + \dots + s_{d-1}x_{d-1} + px_d,$$

where $r_1, \dots, r_{d-1} \in R$; $p, s_1, \dots, s_{d-1} \in I^{n-1}$. Hence $px_d \in (x_1, \dots, x_{d-1})$. Since x_1, \dots, x_{d-1}, x_d is a regular sequence, $p \in (x_1, \dots, x_{d-1})$. Therefore

$$\begin{aligned}
 I^n \cap (x_1, \dots, x_{d-1}) &= (x_1, \dots, x_{d-1})I^{n-1} + x_d(I^{n-1} \cap (x_1, \dots, x_{d-1})) \\
 &= (x_1, \dots, x_{d-1})I^{n-1} + x_d(x_1, \dots, x_{d-1})I^{n-2} \\
 &= (x_1, \dots, x_{d-1})I^{n-1}.
 \end{aligned}$$

Hence x_1^*, \dots, x_{d-1}^* is a $G(I)$ -regular sequence. □

Consequences of Huckaba-Marley Theorem

The characterizations of depth of $G(I)$ in terms of $e_1(I)$ imply several results obtained by various authors. We assume in this section that (R, \mathfrak{m}) is Cohen-Macaulay of dimension d , I is an \mathfrak{m} -primary ideal and R/\mathfrak{m} is infinite.

Corollary 6.3 (Northcott, 1960). $e_1(I) = 0$ if and only if I is generated by a regular sequence.

Proof. For a minimal reduction J of I , $\sum_{n=1}^{\infty} \lambda(I^n/J \cap I^n) \leq e_1(I) = 0$. Therefore $\lambda(I/J) = 0$ which means $I = J$. \square

Corollary 6.4 (Sally, 1980). Let $r(\mathfrak{m}) \leq 2$. Then $G(\mathfrak{m})$ is Cohen-Macaulay.

Proof. For a minimal reduction J of \mathfrak{m} , $J \cap \mathfrak{m}^2 = \mathfrak{m}J$ since a minimal basis of J can be extended to a minimal basis of \mathfrak{m} . Since $\mathfrak{m}^3 = J\mathfrak{m}^2$, for $n \geq 3$, $\mathfrak{m}^n \cap J = J\mathfrak{m}^{n-1} \cap J = J\mathfrak{m}^{n-1}$ and $\mathfrak{m}^n = J\mathfrak{m}^{n-1}$. Therefore

$$\begin{aligned} \sum_{n=1}^{\infty} \lambda\left(\frac{\mathfrak{m}^n + J}{J}\right) &= \lambda(\mathfrak{m}/J) + \lambda(\mathfrak{m}^2/J\mathfrak{m}) \\ &\leq e_1(\mathfrak{m}) \leq \sum_{n=1}^{\infty} \lambda\left(\frac{\mathfrak{m}^n}{J\mathfrak{m}^{n-1}}\right) \\ &= \lambda(\mathfrak{m}/J) + \lambda(\mathfrak{m}^2/J\mathfrak{m}). \end{aligned}$$

Thus $e_1(\mathfrak{m}) = \sum_{n=1}^{\infty} \lambda(\mathfrak{m}^n/J \cap \mathfrak{m}^n)$. Hence $G(\mathfrak{m})$ is Cohen-Macaulay. \square

Corollary 6.5 (Huneke-Ooishi, 1987). $e_0(I) - e_1(I) = \lambda(R/I)$ if and only if $r(I) \leq 1$. In this case $G(I)$ is Cohen-Macaulay and for all $n \geq 1$,

$$\lambda(R/I^n) = e_0(I) \binom{n+d-1}{d} - e_1(I) \binom{n+d-2}{d-1}.$$

Proof. Let $e_0(I) - e_1(I) = \lambda(R/I)$. Let J be any minimal reduction of I . Then $e_0(I) = \lambda(R/J)$ so $e_1(I) = \lambda(I/J) \geq \sum_{n=1}^{\infty} \lambda(I^n/J \cap I^n)$. Therefore $\lambda(I^n/J \cap I^n) = 0$ for all $n \geq 2$. This implies that $G(I)$ is Cohen-Macaulay. Hence $I^2 = JI$. Let $J = (a_1, \dots, a_d)$. Then a_1^*, \dots, a_d^* is a $G(I)$ -regular sequence. Hence

$$\begin{aligned} G(I)/(a_1^*, \dots, a_d^*) &\cong G(I/J) = \bigoplus_{n=0}^{\infty} \frac{I^n + J}{I^{n+1} + J} \\ &= \frac{R}{I} \bigoplus \frac{I}{I^2 + J} \bigoplus \frac{I^2}{I^3 + J} \bigoplus \dots \\ &= \frac{R}{I} \bigoplus \frac{I}{J}. \end{aligned}$$

Therefore

$$\begin{aligned} H(G(I), t) &= \sum_{n=0}^{\infty} \lambda(I^n/I^{n+1})t^n = \frac{h_0 + h_1t + \dots + h_st^s}{(1-t)^d} = \frac{H(G(I/J), t)}{(1-t)^d} \\ &= \frac{\lambda(R/I) + (\lambda(R/J) - \lambda(R/I))t}{(1-t)^d} \end{aligned}$$

Hence $h(t) = \lambda(R/I) + [e_0(I) - \lambda(R/I)]t$ which gives $e_1(I) = e_0(I) - \lambda(R/I)$ and $e_2 = e_3 = \dots = e_d = 0$ by [2, Proposition 4.1.9]. Now we find

$\lambda(R/I^{n+1}) :$

$$\begin{aligned} \left(\sum_{n=0}^{\infty} \lambda(R/I^{n+1}) t^n \right) (1-t) &= \sum_{n=0}^{\infty} \lambda(R/I^{n+1}) t^n - \sum_{n=0}^{\infty} \lambda(R/I^{n+1}) t^{n+1} \\ &= \sum_{n=0}^{\infty} \lambda(I^n/I^{n+1}) t^n. \end{aligned}$$

Hence

$$\begin{aligned} \sum_{n=0}^{\infty} \lambda(R/I^{n+1}) t^n &= \frac{\lambda(R/I) + [e_0(I) - \lambda(R/I)]t}{(1-t)^{d+1}} \\ &= [\lambda(R/I) + (e_0(I) - \lambda(R/I))t] \left(\sum_{n=0}^{\infty} \binom{n+d}{d} t^n \right) \end{aligned}$$

Thus for $n \geq 1$,

$$\begin{aligned} \lambda(R/I^{n+1}) &= \lambda(R/I) \binom{n+d}{d} + [e_0(I) - \lambda(R/I)] \binom{n-1+d}{d} \\ &= e_0(I) \binom{n+d}{d} - [e_0(I) - \lambda(R/I)] \binom{n+d-1}{d-1}. \end{aligned}$$

□

The next result is surprising since it shows that the Cohen-Macaulay property of the Rees algebra $\mathcal{R}(I)$ can be determined by $e_1(I)$ and a minimal reduction of I . Goto and Shimoda [6] showed that $\mathcal{R}(I)$ is Cohen-Macaulay if and only if $G(I)$ is Cohen-Macaulay and $r(I) \leq d-1$.

Corollary 6.6. *Let (R, \mathfrak{m}) be a d -dimensional Cohen-Macaulay local ring with infinite residue field. Let J be a minimal reduction of I . Then $\mathcal{R}(I)$ is Cohen-Macaulay if and only if $e_1(I) = \sum_{n=1}^{d-1} \lambda(I^n/J \cap I^n)$.*

Proof. Let $\mathcal{R}(I)$ be Cohen-Macaulay. Hence $G(I)$ is Cohen-Macaulay and $r(I) \leq d-1$. Therefore for any minimal reduction J of I , $J I^n = I^{n+1} = J \cap I^n$ for all $n \geq d-1$. Thus $e_1(I) = \sum_{n=1}^{d-1} \lambda(I^n/J \cap I^n)$. Conversely, if $e_1(I) = \sum_{n=1}^{d-1} \lambda(I^n/J \cap I^n)$, then by the inequality $e_1(I) \geq \sum_{n=1}^{\infty} \lambda(I^n/J \cap I^n)$ we get $I^n = J \cap I^n$ for all $n \geq d$ which implies $I^n \subseteq J$ for all $n \geq d$. Moreover, $e_1(I) = \sum_{n=1}^{\infty} \lambda(I^n/J \cap I^n)$. Hence $G(I)$ is Cohen-Macaulay. Therefore $I^d \cap J = J I^{d-1} = I^d$. By Goto-Shimoda Theorem, $\mathcal{R}(I)$ is Cohen-Macaulay. □

The next theorem due to A. Guerriere [5] gives a condition in terms of minimal reduction, for the depth $G(I) = d-1$. The original proof of this was rather involved.

Corollary 6.7 (A. Guerriere, 1994). *Suppose $\sum_{n \geq 2} \lambda(J \cap I^n / J I^{n-1}) = 1$. Then $\text{depth } G(I) = d-1$.*

Proof. We know that

$$\sum_{n=1}^{\infty} \lambda(I^n/J \cap I^n) \leq e_1(I) \leq \sum_{n=1}^{\infty} \lambda(I^n/JI^{n-1}).$$

Since $\sum_{n=1}^{\infty} \lambda(I^n/JI^{n-1}) - \sum_{n=1}^{\infty} \lambda(I^n/J \cap I^n) = 1$ we conclude $e_1(I) = \sum_{n=1}^{\infty} \lambda(I^n/J \cap I^n)$ or $e_1(I) = \sum_{n=1}^{\infty} \lambda(I^n/JI^{n-1})$. The former case does not occur since otherwise $G(I)$ will be Cohen-Macaulay and consequently $I^n \cap J = JI^{n-1}$ for all $n \geq 1$. Hence $\sum_{n=1}^{\infty} \lambda(J \cap I^n/JI^{n-1}) = 0$, which is a contradiction. Therefore $\text{depth } G(I) = d - 1$. \square

REFERENCES

- [1] S. S. Abhyankar, *Local rings of high embedding dimension*, Amer. J. Math. **89** (1967), 1073-1077.
- [2] W. Bruns and J. Herzog, *Cohen-Macaulay Rings*, Revised Edition, Cambridge University Press, 1998.
- [3] A. Corso, C. Polini and M. Vaz Pinto, *Sally modules and associated graded rings*, Communications in algebra **26** (1998), 2689-2708.
- [4] J. Elias, *On the depth of the tangent cone and the growth of the Hilbert function*, Trans. Amer. Math. Soc. **351** (199), 4027-4042.
- [5] A. Guerrieri, *On the depth of the associated graded ring of an \mathfrak{m} -primary ideal of a Cohen-Macaulay local ring*, J. Algebra **167** (1994), 745-757.
- [6] S. Goto and Y. Shimoda, *On the Rees algebras of Cohen-Macaulay local rings*, Commutative algebra, Lecture Notes in Pure and Appl. Math., 68, Dekker, New York (1982), 201-231.
- [7] C. Huneke, *Hilbert functions and symbolic powers*, Michigan Math. J. **34** (1987), 293-318.
- [8] S. Huckaba, *A d -dimensional extension of a lemma of Huneke's and formulas for Hilbert coefficients*, Proc. Amer. Math. Soc. **124** (1996), 1393-1401.
- [9] S. Huckaba and T. Marley, *Hilbert coefficients and the depths of associated graded rings*, J. London Math. Soc. (2) **56** (1997), 64-76.
- [10] C. Huneke and I. Swanson, *Integral closure of ideals, rings and modules*, Cambridge University Press, 2006.
- [11] A. V. Jayanthan, B. Singh and J. K. Verma, *Hilbert coefficients and depths of form rings*, Comm. Algebra, **32** (2004), 1445-1452.
- [12] M. P. Murthy, *Commutative Algebra Vol 1 and II*, University of Chicago Lecture Notes, 1976.
- [13] M. Nagata, *Local Rings*, Interscience, 1962.
- [14] D. G. Northcott, *A note on the coefficients of the abstract Hilbert function*, J. London Math. Soc. **35** (1960), 209-214.
- [15] D. G. Northcott and D. Rees, *Reductions of ideals in local rings*, Math. Proc. Cambridge Phil. Soc. **50** (1954), 145-158.
- [16] A. Ooishi, *Δ -genera and sectional genera of commutative rings*, Hiroshima Math. J. **17** (1987), 361-372.
- [17] M. E. Rossi, *A bound on the reduction number of a primary ideal*, Proc. Amer. Math. Soc. **128** (2000), 1325-1332.
- [18] M. E. Rossi and G. Valla, *A conjecture of J. Sally*, Communications in Algebra **24** (1996), 4249-4261.

- [19] J. D. Sally, *On the associated graded ring of a local Cohen-Macaulay ring*, J. Math Kyoto U. **17** (1977), 19-21.
- [20] J. D. Sally, *Tangent cones at gorenstein singularities*, Comp. Math. **40** (1980), 167-175.
- [21] J. D. Sally, *Cohen-Macaulay local rings of embedding dimension $e + d - 2$* , J. Algebra **83** (1983), 393-408.
- [22] G. Valla, *Problems and results on Hilbert functions of graded algebras*, Six Lectures on Commutative Algebra, Edited by J. Elias, J. M. Giral, R. M. Miro'-Roig and S. Zarzuela, Birkh'auser Verlag, 1998.
- [23] P. Valabrega and G. Valla, *Form rings and regular sequences*, Nagoya Math. Journal **72** (1978), 93-101.
- [24] W. Vasconcelos, *Hilbert functions, analytic spread and Koszul homology*, Contemporary Mathematics **159** (1994), 401-422.
- [25] M. Vaz Pinto, *Hilbert functions and Sally modules*, Journal of Algebra **192** (1996), 504-523.
- [26] H.-J. Wang, *On Cohen-Macaulay local rings with embedding dimension $e+d-2$* , J. Algebra **190** (1997), 226-240.

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